

Turán numbers of bipartite graphs plus an odd cycle

Peter Allen^{*} Peter Keevash[†] Benny Sudakov[‡] Jacques Verstraëte[§]

Abstract

For an odd integer k , let $\mathcal{C}_k = \{C_3, C_5, \dots, C_k\}$ denote the family of all odd cycles of length at most k and let \mathcal{C} denote the family of all odd cycles. Erdős and Simonovits [10] conjectured that for every family \mathcal{F} of bipartite graphs, there exists k such that $\text{ex}(n, \mathcal{F} \cup \mathcal{C}_k) \sim \text{ex}(n, \mathcal{F} \cup \mathcal{C})$ as $n \rightarrow \infty$. This conjecture was proved by Erdős and Simonovits when $\mathcal{F} = \{C_4\}$, and for certain families of even cycles in [13]. In this paper, we give a general approach to the conjecture using Scott's sparse regularity lemma. Our approach proves the conjecture for complete bipartite graphs $K_{2,t}$ and $K_{3,3}$: we obtain more strongly that for any odd $k \geq 5$,

$$\text{ex}(n, \mathcal{F} \cup \{C_k\}) \sim \text{ex}(n, \mathcal{F} \cup \mathcal{C})$$

and we show further that the extremal graphs can be made bipartite by deleting very few edges. In contrast, this formula does not extend to triangles – the case $k = 3$ – and we give an algebraic construction for odd $t \geq 3$ of $K_{2,t}$ -free C_3 -free graphs with substantially more edges than an extremal $K_{2,t}$ -free bipartite graph on n vertices. Our general approach to the Erdős-Simonovits conjecture is effective based on some reasonable assumptions on the maximum number of edges in an m by n bipartite \mathcal{F} -free graph.

1 Introduction

Let \mathcal{F} be a family of graphs. Then we say that a graph is \mathcal{F} -free if it contains no member of \mathcal{F} as a (not necessarily induced) subgraph. We define the *Turán number* $\text{ex}(n, \mathcal{F})$ to be the maximum number of edges possible in a \mathcal{F} -free graph on n vertices. When $\mathcal{F} = \{F\}$ consists of a single forbidden graph we denote the Turán number by $\text{ex}(n, F)$. A classical theorem of Turán [25] gives an exact result for $\text{ex}(n, K_r)$, where K_r is the complete graph on r vertices: the unique largest K_r -free graph on n vertices is the complete $(r-1)$ -partite graph with part sizes as equal as possible. In general, Erdős, Stone and Simonovits [11, 8] showed that $\text{ex}(n, \mathcal{F}) \sim (1 - 1/r) \binom{n}{2}$, where $r = \min\{\chi(F) - 1 : F \in \mathcal{F}\}$. Here we write $f(n) \sim g(n)$ for functions $f, g : \mathbb{N} \rightarrow \mathbb{R}$ if $\lim_{n \rightarrow \infty} f(n)/g(n) = 1$. This determines the Turán number asymptotically when \mathcal{F} consists only of non-bipartite graphs. Furthermore, Erdős and Simonovits [9] proved that the error term is zero if the family \mathcal{F} contains an edge-critical $(r+1)$ -chromatic graph (i.e. one from which one edge can be removed to decrease the chromatic number).

^{*}Department of Mathematics, London School of Economics, UK, p.d.allen@lse.ac.uk. Research supported in part by FAPESP (Proc. 2010/09555-7).

[†]School of Mathematical Sciences, Queen Mary University of London, UK, p.keevash@qmul.ac.uk. Research supported in part by ERC grant 239696 and EPSRC grant EP/G056730/1.

[‡]Department of Mathematics, University of California, Los Angeles, CA, USA, bsudakov@math.ucla.edu. Research supported in part by NSF grant DMS-1101185, by AFOSR MURI grant FA9550-10-1-0569 and by a USA-Israel BSF grant.

[§]Department of Mathematics, University of California, San Diego, CA, USA, jverstra@math.ucsd.edu. Research supported by NSF Grant DMS-110 1489.

For more general families \mathcal{F} the error term is necessary, and the complete balanced r -partite graph is not extremal. In that case, it turns out that the error term is controlled by Turán numbers for bipartite graphs [22]. The major open problem in the area, then, is to determine the behavior of $\text{ex}(n, \mathcal{F})$ for families \mathcal{F} including bipartite graphs.

1.1 Turán Numbers for Bipartite Graphs

The problem of determining $\text{ex}(n, \mathcal{F})$ when \mathcal{F} contains bipartite graphs seems in general to be very hard: in almost all cases we do not know the order of magnitude of $\text{ex}(n, \mathcal{F})$. In a seminal paper [10], Erdős and Simonovits made a number of broad conjectures on Turán numbers for bipartite graphs. In particular (see Conjecture 2 in [10]) they made the following conjecture:

Conjecture 1.1. *For every finite nonempty family \mathcal{F} of graphs, there exist $\sigma, \alpha \geq 0$ such that*

$$\lim_{n \rightarrow \infty} \frac{\text{ex}(n, \mathcal{F})}{n^\alpha} = \sigma.$$

Erdős and Simonovits [10] stated that α should furthermore be rational, in agreement with all the cases where α is known to exist. When α exists, we refer to α as the *exponent* of \mathcal{F} . The lack of good constructive lower bounds for $\text{ex}(n, \mathcal{F})$ is a major impediment to determining exponents of families of bipartite graphs. It appears unlikely that the order of magnitude of $\text{ex}(n, \mathcal{F})$, let alone the existence of the exponent of \mathcal{F} , will ever be determined for every bipartite graph \mathcal{F} . Even for specific graphs such as the complete bipartite graph $F = K_{4,4}$, the cycle of length eight $F = C_8$, and the three-dimensional cube graph $F = Q_3$, the order of magnitude of $\text{ex}(n, F)$ is not known.

1.2 A conjecture of Erdős and Simonovits

In this paper, we study a conjecture of Erdős and Simonovits [10] concerning Turán numbers for families \mathcal{F} containing bipartite graphs and odd cycles. Let \mathcal{C}_k be the family of all odd cycles of length at most k , and let $z(n, \mathcal{F})$ denote the maximum size of a bipartite n -vertex \mathcal{F} -free graph. The general theme is that extremal \mathcal{F} -free graphs should be near-bipartite if \mathcal{F} contains a long enough odd cycle as well as bipartite graphs. Precisely, Erdős and Simonovits made the following conjecture (Conjecture 3 in [10]):

Conjecture 1.2. *Given any finite family \mathcal{F} of graphs, there exists k such that as $n \rightarrow \infty$,*

$$\text{ex}(n, \mathcal{F} \cup \mathcal{C}_k) \sim z(n, \mathcal{F}).$$

This is certainly true from the Erdős-Stone-Simonovits Theorem [9] when \mathcal{F} consists only of non-bipartite graphs, moreover it is shown that the extremal $\mathcal{F} \cup \{C_k\}$ -free graphs are exactly bipartite if n is large enough [9]. Conjecture 1.2 is proved in [13] for $\mathcal{F} = \{C_4, C_6, \dots, C_{2\ell}\}$ when $\ell \in \{2, 3, 5\}$, and again the extremal graphs are shown in [13] to be bipartite graphs: the bipartite incidence graphs of rank two geometries called generalized polygons [24]. The case $\mathcal{F} = \{C_4\}$ with $k = 5$ was settled earlier by Erdős and Simonovits [10].

In the present paper, we give a general approach which proves the conjecture for all families of bipartite graphs which are known to be *smooth* (in a sense which is defined below) using a sparse version of

Szemerédi's Regularity Lemma. It is convenient for the next definition to denote by $z(m, n, \mathcal{F})$ the maximum number of edges in an m by n bipartite \mathcal{F} -free graph. In what follows, all asymptotic notation is with respect to n . Let α, β be reals with $1 \leq \beta < \alpha < 2$.

Definition 1. A family \mathcal{F} of bipartite graphs is (α, β) -smooth if for some $\rho \geq 0$ and every $m \leq n$,

$$z(m, n, \mathcal{F}) = \rho mn^{\alpha-1} + O(n^\beta).$$

It is a consequence of the Kővári-Sós-Turán Theorem [17] that $\alpha < 2$ for every family \mathcal{F} of bipartite graphs. We say \mathcal{F} is *smooth* if \mathcal{F} is (α, β) -smooth for some $\alpha > \beta$, and (α, β) -smooth with *relative density* ρ when we wish to refer to the limit density. If \mathcal{F} is smooth and consists of a single graph F , then we use the terminology *F is smooth*. Conjecture 1.1 perhaps suggests that every finite family of graphs is smooth – smoothness should be seen as a bipartite version of Conjecture 1.1 together with some control on the error term implicit in the limit. In this paper, we study smooth families of graphs and use smoothness as a broad framework under which Conjecture 1.2 might be proved. Single bipartite graphs which are known to be smooth include $K_{2,t}$ for $t \geq 2$ and $K_{3,3}$, however, even the cycle C_6 of length six is not known to be smooth.

1.3 Main Result

A family \mathcal{G} of graphs is *near-bipartite* if every n -vertex graph $G \in \mathcal{G}$ has a bipartite subgraph H with $e(G) \sim e(H)$ as $n \rightarrow \infty$. To prove Conjecture 1.2 for a family \mathcal{F} , it is enough to show every extremal $\mathcal{F} \cup \{C_k\}$ -free family of graphs is near-bipartite for some odd k , for then $\text{ex}(n, \mathcal{F} \cup \{C_k\}) \sim z(n, \mathcal{F})$. Our main theorem proves this strengthening of the Erdős-Simonovits conjecture for smooth families of bipartite graphs:

Theorem 1.3. *Let \mathcal{F} be an (α, β) -smooth family with $2 > \alpha > \beta \geq 1$. There exists k_0 such that if $k \geq k_0 \in \mathbb{N}$ is odd, then every extremal $\mathcal{F} \cup \{C_k\}$ -free family of graphs is near-bipartite.*

The theorem settles Conjecture 1.2 in a strong sense for smooth families, namely that the omission of any single odd cycle is enough to force extremal \mathcal{F} -free families of graphs to be nearly bipartite, whereas the condition of Conjecture 1.2 is to avoid all short odd cycles. The closer the value of β to α , however, the larger the lower bound on k , the length of the shortest odd cycle we can find. The proof of Theorem 1.3 is a novel application of a recent sparse version of Szemerédi's Regularity Lemma due to Scott [20]. We note that (despite the use of regularity) the value of k_0 we obtain is not large – for all the smooth families we know, it is less than 20. We also ask whether the extremal n -vertex $\mathcal{F} \cup \{C_k\}$ -free graphs in Theorem 1.3 are exactly bipartite when n is large enough. So far this is not known, even for $\mathcal{F} = \{K_{2,t}\}$ with $t > 2$, which is $(3/2, 4/3)$ -smooth. Here we give some evidence that extremal $\mathcal{F} \cup \{C_k\}$ -free graphs are indeed bipartite, by proving the following Andrásfai-Erdős-Sós type theorem, which verifies this provided the minimum degree of an extremal graph is not too small.

Theorem 1.4. *Let \mathcal{F} be an (α, β) -smooth family with $2 > \alpha > \beta \geq 1$. Then there exists k_0 such that for all odd $k \geq k_0$ and n sufficiently large the following is true. If G is an n -vertex $\mathcal{F} \cup \{C_k\}$ -free graph with minimum degree at least $\rho(\frac{2n}{5} + o(n))^{\alpha-1}$ then G is bipartite.*

1.4 Complete Bipartite Graphs

Theorem 1.3 reduces the Erdős-Simonovits Conjecture to showing that a family of bipartite graphs is smooth. We shall now see that certain complete bipartite graphs are smooth. Füredi [12] showed that if $m \leq n$, and $s, t \in \mathbb{N}$, then

$$z(m, n, K_{s,t}) \leq (t - s + 1)^{1/s} mn^{1-1/s} + sm + sn^{2-2/s}.$$

If $s = 2$ or $t = s = 3$, this upper bound is matched up to a difference of $O(n^{2-2/s})$ by constructions of Brown [4] and Füredi [12], so in these cases, $K_{s,t}$ is $(2 - 1/s, 2 - 2/s)$ -smooth. It follows from Theorem 1.3 that the Erdős-Simonovits conjecture holds for the complete bipartite graphs $K_{2,t}$ and $K_{3,3}$, and we obtain the following result:

Theorem 1.5. *Let $k \geq 5$ be an odd integer. If $s = 2$ and $t \geq 2$, or $s = t = 3$, then*

$$\text{ex}(n, \{K_{s,t}, C_k\}) \sim z(n, K_{s,t}) \sim (t - s + 1)^{1/s} (n/2)^{2-1/s}$$

and any extremal \mathcal{F} -free C_k -free graph is near-bipartite.

This theorem is a direct consequence of Theorem 1.3 and the smoothness of $K_{s,t}$ for $s = 2$ and $s = t = 3$, except that the assumption $k \geq 5$ is weaker than the assumption on k in Theorem 1.3. For $t > (s - 1)!$, the order of magnitude of $\text{ex}(n, K_{s,t})$ is known to be $n^{2-1/s}$ according to a construction of Alon, Rónyai and Szabó [3], superseding an earlier construction of Kollár, Rónyai and Szabó [15]. However, the asymptotic behavior of $z(m, n, K_{s,t})$ is not known for any $s \geq 3$ and $t \geq 4$, and the order of magnitude of $\text{ex}(n, K_{s,t})$ is not known in general. In the event that $K_{s,t}$ is $(2 - 1/s, \beta)$ -smooth for any given $s \leq t$ and $\beta < 2 - 1/s$, Theorem 1.5 would extend to prove the Erdős-Simonovits conjecture for $K_{s,t}$ with the condition $k \geq 5$.

1.5 Triangles and an old conjecture of Erdős

A natural question is whether it is also possible that Theorems 1.3 and 1.5 could hold for shorter odd cycles, in particular for triangles – the case $k = 3$ in both theorems. In the case $s = t = 2$, it is an old conjecture of Erdős [6, 7] that any extremal n -vertex graph of girth five has $\frac{1}{2\sqrt{2}}n^{3/2} + o(n^{3/2})$ edges – in particular the bipartite incidence graph of a projective plane with n points has that many edges. However, we are able to show that for all odd $t \geq 3$, the extremal function $\text{ex}(n, \{K_{2,t}, C_3\})$ is not asymptotic to $z(n, K_{2,t})$.

Theorem 1.6. *For each $t \in \mathbb{N}$,*

$$\liminf_{n \rightarrow \infty} \frac{\text{ex}(n, \{C_3, K_{2,2t+1}\})}{z(n, K_{2,2t+1})} \geq \frac{t+1}{\sqrt{t(t+2)}}.$$

In particular the ratio is $\frac{2}{\sqrt{3}} + o(1)$ for $K_{2,3}$. In light of this, we make the opposite conjecture to Erdős:

Conjecture 1.7.

$$\liminf_{n \rightarrow \infty} \frac{\text{ex}(n, \{C_3, C_4\})}{z(n, C_4)} > 1.$$

In [13], it is shown that Conjecture 1.2 does not extend to $k = 3$ when $\mathcal{F} = \{C_4, C_6\}$ and does not extend to $k = 3, 5$ when $\mathcal{F} = \{C_4, C_6, C_8, C_{10}\}$. Determining the smallest value of $k = k(\mathcal{F})$ for which the conjecture could hold is an interesting open problem.

2 Sparse regularity

In this section we describe a version of Szemerédi's Regularity Lemma which can be usefully applied to extremal \mathcal{F} -free graphs.

2.1 Szemerédi's Regularity Lemma

Given an n -vertex graph G and disjoint vertex subsets X and Y , the *density* of the pair (X, Y) is $d(X, Y) = e(X, Y)/(|X||Y|)$. We say that (X, Y) is ε -regular if we have $|d(X', Y') - d(X, Y)| \leq \varepsilon$ for every pair $X' \subset X$ and $Y' \subset Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$. We say a partition $V(G) = V_0 \sqcup \dots \sqcup V_k$ is ε -regular if $|V_0| \leq \varepsilon n$, $|V_1| = \dots = |V_k|$, and all but at most εk^2 of the pairs (V_i, V_j) with $i, j \in [k]$ are ε -regular. The celebrated Szemerédi Regularity Lemma [23] states that for every $\varepsilon > 0$ there exists K such that every graph G possesses an ε -regular partition whose number of parts is between ε^{-1} and K . A key definition in the proof is the *energy function* of a partition \mathcal{P} of $V(G)$, which is

$$\mathcal{E}(G, \mathcal{P}) = \sum_{X, Y \in \mathcal{P}} \frac{|X||Y|}{n^2} d(X, Y)^2. \quad (1)$$

The proof consists of showing that either \mathcal{P} is an ε -regular partition of G , or there is a refinement of \mathcal{P} with at most $2^{|\mathcal{P}|}$ parts whose energy is increased by at least ε^5 over that of \mathcal{P} . Since the energy of any partition is bounded by 1, the result follows. Unfortunately, Szemerédi's Regularity Lemma in the above framework is useless for our purposes: $K_{s,t}$ -free graphs have $o(n^2)$ edges, and hence the densities of all pairs in the partition will be very close to zero.

2.2 Versions of the regularity lemma for sparse graphs

For sparse graphs, Kohayakawa [14] and independently Rödl (unpublished) defined the notion of relative density and (ε, p) -regular partitions where $p \in [0, 1]$. Here, G is a graph and X and Y are disjoint non-empty subsets of vertices of G .

Definition 2. Given $0 < p \leq 1$, the relative density of the pair (X, Y) is $d_p(X, Y) = d(X, Y)/p$. The pair (X, Y) is (ε, p) -regular if $|d_p(X', Y') - d_p(X, Y)| \leq \varepsilon$ for every pair $X' \subset X$ and $Y' \subset Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$. A partition $V(G) = V_0 \sqcup \dots \sqcup V_k$ is called (ε, p) -regular if $|V_0| \leq \varepsilon n$, $|V_1| = \dots = |V_k|$, and all but at most εk^2 of the pairs (V_i, V_j) with $i, j \in [k]$ are (ε, p) -regular.

Kohayakawa [14] and Rödl proved that for any function $p = p(n)$, $\varepsilon > 0$ and C there exist K and $\mu > 0$ such that if G is an n -vertex graph for which the relative density $d_p(X, Y)$ is at most C whenever X and Y are disjoint vertex sets of size at least μn , then G has an (ε, p) -regular partition whose number of parts is between ε^{-1} and K . For this proof, the energy function

$$\mathcal{E}_p(G, \mathcal{P}) = \sum_{X, Y \in \mathcal{P}} \frac{|X||Y|}{n^2} d_p(X, Y)^2 \quad (2)$$

is defined, and is observed to be bounded by C whenever \mathcal{P} contains no small parts; the remainder of the proof is virtually identical to the original proof of Szemerédi. Unfortunately, $K_{s,t}$ -free graphs with $\Theta(n^{2-1/s})$ edges and $p(n) = n^{-1/s}$ need not satisfy the condition given by Kohayakawa and Rödl. For complete bipartite graphs this is not a major issue: one can use instead the following bound on the energy function of a partition, from which the desired regularity lemma follows.

Lemma 2.1. *Let $p = n^{-1/s}$. Let G be a $K_{s,t}$ -free graph and \mathcal{P} a partition of $V(G)$ with no parts of size smaller than $2sn^{1/s}$. Then we have $\mathcal{E}_p(G, \mathcal{P}) \leq 2^s t + 1$.*

Proof. Let G be an n -vertex $K_{s,t}$ -free graph, and \mathcal{P} a partition of $V(G)$ with no parts containing fewer than $2sn^{1/s}$ vertices. Given part $X \in \mathcal{P}$ we have

$$\frac{t|X|^s}{s!} \geq t \binom{|X|}{s} \geq \sum_{v \in G} \binom{d_X(v)}{s} \geq \sum_{Y \in \mathcal{P}} |Y| \binom{d(X, Y)|X|}{s} = \sum_{Y \in \mathcal{P}} |Y| \binom{d_p(X, Y)n^{-1/s}|X|}{s},$$

using that G is $K_{s,t}$ -free in the second step and Jensen's inequality at the third step. Observe that since $|X| > 2sn^{1/s}$, if $d_p(X, Y) \geq 1$ then $d_p(X, Y)n^{-1/s}|X| - s \geq d_p(X, Y)n^{-1/s}|X|/2$. Ignoring terms with $d_p(X, Y) < 1$, we have

$$\frac{t|X|^s}{s!} \geq \sum_{\substack{Y \in \mathcal{P} \\ d_p(X, Y) \geq 1}} |Y| \binom{d_p(X, Y)n^{-1/s}|X|}{s} \geq \sum_{\substack{Y \in \mathcal{P} \\ d_p(X, Y) \geq 1}} \frac{|Y| d_p(X, Y)^s |X|^s}{n 2^s s!}$$

and thus

$$\sum_{\substack{Y \in \mathcal{P} \\ d_p(X, Y) \geq 1}} \frac{|Y|}{n} d_p(X, Y)^s \leq 2^s t.$$

Finally, since $x^2 \leq x^s$ for all $x \geq 1$ and $x^2 < 1$ for all $x < 1$, we have

$$\begin{aligned} \mathcal{E}_p(G, \mathcal{P}) &= \sum_{X, Y \in \mathcal{P}} \frac{|X||Y|}{n^2} d_p(X, Y)^2 \leq \sum_{X \in \mathcal{P}} \frac{|X|}{n} \left(\sum_{\substack{Y \in \mathcal{P} \\ d_p(X, Y) \geq 1}} \frac{|Y|}{n} d_p(X, Y)^s + \sum_{\substack{Y \in \mathcal{P} \\ d_p(X, Y) < 1}} \frac{|Y|}{n} \right) \\ &\leq \sum_{X \in \mathcal{P}} \frac{|X|}{n} (2^s t + 1) = 2^s t + 1 \end{aligned}$$

as required. □

However, Scott [20] has recently shown that (perhaps surprisingly) we need assume no conditions at all on G to obtain a sparse-regular partition, and it is convenient to simply use this result to prove our main theorems. To the best of our knowledge, this is the first application of Scott's regularity lemma.

Lemma 2.2. (Sparse Regularity Lemma) *Given $\varepsilon > 0$ and $C \geq 1$, there exists $T = T(\varepsilon)$ such that if G is any graph with at most Cpn^2 edges, then G has an (ε, p) -regular partition whose number of parts is between ε^{-1} and T .*

Briefly, Scott's proof rests on his observation that the Kohayakawa-Rödl proof fails for general graphs because it is possible for $\mathcal{E}_p(G, \mathcal{P})$ to be large due to a small number of dense pairs in \mathcal{P} , and thus no good enough bound on \mathcal{E}_p for that proof to work can hold. Scott's solution is to define a new energy function, replacing $d_p(X, Y)^2$ with $\min(d_p(X, Y)^2, 4L(d_p(X, Y) - L))$ for some large constant L . This has the effect of 'discounting' very dense pairs, so that the new energy function can be proven to be bounded (by CL^2): of course, it also means that refining these very dense pairs in a non- (ε, p) -regular partition no longer contributes to increasing the energy function, but, crucially, there are few such dense pairs and thus an energy increase of $\varepsilon^5/2$ is still guaranteed. To facilitate use of Lemma 2.2, we give one last definition.

Definition 3. *Given $d \in [0, 1]$, the (ε, d, p) -cluster graph associated to an (ε, p) -regular partition $V_0 \sqcup \dots \sqcup V_t$ of G is the graph with vertex set $\{V_1, \dots, V_t\}$ and edges $V_i V_j$ whenever (V_i, V_j) is an (ε, p) -regular pair with $d_p(V_i, V_j) \geq d$.*

We shall also require some basic useful facts about regular pairs. These are proved directly from the definition of regularity and density, and are standard facts in the usual dense version of the regularity lemma (see [16]).

Proposition 2.3. *Let (A, B) and (A, C) be (ε, p) -regular pairs of relative density at least d . Then*

1. *$(A, B \cup C)$ is (ε, p) -regular of relative density at least d .*
2. *If $\gamma > \varepsilon$ and $X \subseteq B$ has size at least $\gamma|B|$, then (A, X) is $(\varepsilon + \varepsilon/\gamma, p)$ -regular of relative density at least $d - \varepsilon$.*

3 Expansion and embedding

In this section, we show how to use the presence of many triangles in an (ε, d, p) -cluster graph of an \mathcal{F} -free graph G to find odd cycles in G . In the traditional setting of the regularity lemma, namely for dense graphs, extremal theorems in the cluster graph are employed to find large structures in the original graph. In particular, a single clique K_r in the cluster graph is sufficient to provide a relatively large complete r -partite subgraph of the original graph, which gives a derivation of the Erdős-Stone Theorem from Turán's Theorem via the regularity lemma. However, in the case of sparse graphs, the appearance of a particular graph H in the (ε, d, p) -cluster graph is insufficient to imply a copy of H in the original graph. For example, a construction of Alon [2] shows that we cannot hope to find even one triangle in a triple of sparse-regular pairs by making use only of the sparse-regularity. Similarly, our example of a dense $K_{2,3}$ -free triangle-free graph in Theorem 1.6 is actually a triple of sparse-regular pairs, so the cluster graph is actually a triangle, whereas the original graph has no triangles as well as no $K_{2,3}$. More generally (see Krivelevich and Sudakov [18]), one can take any pseudorandom C_k -free graph with k odd and replace each vertex v with a set S_v of $o(n)$ new vertices and each edge uv with the complete bipartite graph consisting of all edges between S_u and S_v . This still gives a pseudorandom C_k -free graph, which in particular has a complete cluster graph, whilst the original graph has no C_k . The issue of what further conditions one should place in order to avoid these 'bad examples' is an important question. Recently Conlon, Fox and Zhao [5] made substantial progress in a quite general setting, but our problem is not covered by their work.

Our strategy to find odd cycles is then the following. We will see that, under the assumption of smoothness, \mathcal{F} -free graphs which are close to extremal (in the sense of having the right order of

magnitude number of edges) have reasonably strong expansion properties. This will allow us to find a vertex v such that the set $N_\ell(v)$ of vertices at distance ℓ from v has linear size, and in the event that the cluster graph contains many triangles we can further ensure that this set has substantial intersection with both sides of a dense regular pair. By regularity there is then an edge e in the set, and thus a cycle of length $2\ell + 1$ in the graph (here we ignore the possibility that the two paths from the ends of e to v might intersect, but this is not difficult to resolve).

3.1 Expansion and embedding from smoothness

The smoothness of a family \mathcal{F} of bipartite graphs with exponent α implies expansion in any \mathcal{F} -free graph with large minimum degree. Precisely, we have the following lemma.

Lemma 3.1. *Let \mathcal{F} be an (α, β) -smooth family of bipartite graphs with $2 > \alpha > \beta \geq 1$. Then there exists $C > 1$ such that for any $\delta > 0$, $\gamma = \delta/2C$ and any \mathcal{F} -free bipartite graph $G(U, V)$ on at most $n \geq (1/\gamma)^{1/(\alpha-\beta)}$ vertices, if every vertex in U has degree at least $\delta n^{\alpha-1}$, then*

1. *if $|U| < |V|$, then $|V| \geq \min(\gamma^{1/\beta}|U|^{1/\beta}n^{(\alpha-1)/\beta}, \gamma^{1/(\alpha-1)}n)$.*
2. *if $|U| \geq |V|$, then $|V| \geq \max(\gamma n^{\alpha-1}|U|^{2-\alpha}, \gamma^{1/(\alpha-1)}n)$.*

Proof. If $|U| < |V|$, then since \mathcal{F} is smooth, there exists C (which we may insist is larger than 1) such that

$$\delta n^{\alpha-1}|U| \leq e(U, V) \leq C(|U||V|^{\alpha-1} + |V|^\beta) \leq 2C \max(|U||V|^{\alpha-1}, |V|^\beta).$$

This immediately implies part 1. If $|U| \geq |V|$, then

$$\delta n^{\alpha-1}|U| \leq e(U, V) \leq C(|V||U|^{\alpha-1} + |U|^\beta). \quad (3)$$

Since $n^{\alpha-\beta} \geq 1/\gamma$ and $|U| \leq n$ and $\beta \geq 1$,

$$n^{\alpha-1} \geq \frac{1}{\gamma} n^{\beta-1} \geq \frac{2C}{\delta} |U|^{\beta-1}.$$

We conclude $2C|U|^\beta \leq \delta n^{\alpha-1}|U|$. Therefore (3) gives:

$$\delta n^{\alpha-1}|U| \leq 2C|V||U|^{\alpha-1}.$$

This immediately gives $|V| \geq \gamma n^{\alpha-1}|U|^{2-\alpha}$. By substituting $|U| \geq |V|$, and recalling $1 < \alpha < 2$, we also obtain

$$\delta n^{\alpha-1}|V|^{2-\alpha} \leq 2C|V|$$

which shows $|V| \geq \gamma^{1/(\alpha-1)}n$, as required in part 2 of the lemma. \square

One might reasonably think that in order to embed odd cycles, it will be enough to find one triangle ABB' in the cluster graph. One would select a typical vertex v in A , which has at least $dp|B|/2$ neighbors in B , and similarly in B' : then Lemma 3.1 (applied on BB') guarantees a larger second neighborhood of v in B' , and similarly in B : after k steps we have subsets of $N_k(v)$ in B and B' of linear size which we control by regularity. Unfortunately, the expansion of \mathcal{F} -free graphs is not quite strong enough for this argument to work: the minimum degree depends upon the number of

clusters, and is thus potentially much smaller than $\varepsilon n^{\alpha-1}$. The result is that we can only guarantee that $N_k(v)$ covers a fraction of a cluster which is $o(\varepsilon)$, and (ε, p) -regularity tells us nothing about such small sets. Therefore we use an alternative approach, demanding not one but a positive density of triangles through A in the cluster graph. These triangles can be used for an expansion argument in a large fraction of the graph, independent of the number of parts. We have then the following embedding lemma.

Lemma 3.2. *Let $\tau, d > 0$, $\alpha > \beta > 1$, and let \mathcal{F} be an (α, β) -smooth family of bipartite graphs. Let $\ell_0(\alpha, \beta)$ be*

$$\ell_0 = \left\lfloor \log_{\beta} \left(\frac{(2\beta - \beta^2)(\alpha - 1)}{\alpha - \beta} \right) \right\rfloor \quad \text{for } \beta > 1 \text{ and } \ell_0 = \lfloor 1/(\alpha - 1) \rfloor \quad \text{for } \beta = 1.$$

Let $p = p(n) = n^{\alpha-2}$ for $n \in \mathbb{N}$. Then there exist $\varepsilon_0 = \varepsilon_0(\alpha, \tau, d)$ and $n_0 = n_0(\alpha, \beta, \tau, d)$ such that the following holds. If $\varepsilon < \varepsilon_0$, $n > n_0$, and G is an \mathcal{F} -free graph with an (ε, d, p) -cluster graph R with $t \geq \varepsilon^{-1}$ clusters in which one cluster A is in at least τt^2 triangles, then G contains C_k for each odd $k \in \mathbb{N}$ satisfying

$$2\ell_0 + 5 \leq k \leq \varepsilon d n^{\alpha-1} / 4t.$$

Proof. Our first task is to find a structure in R suitable for our expansion method. This will consist of clusters A, B, B' , sets of clusters \mathcal{C} and \mathcal{C}' , and clusters D_1, \dots, D_{ℓ_0} and D'_1, \dots, D'_{ℓ_0} , which are adjacent as follows. The clusters D_1, \dots, D_{ℓ_0} form a path (in that order); D_1 is adjacent to A , D_{ℓ_0} to all clusters \mathcal{C} , and all clusters \mathcal{C} to B . We require similar adjacencies involving $B', \mathcal{C}', D'_1, \dots, D'_{\ell_0}$. Finally B and B' are adjacent. Our embedding will proceed as follows. We will choose a typical vertex v in A , which then has a substantial first neighborhood in both D_1 and D'_1 . We apply Lemma 3.1 between D_i and D_{i+1} for each i to find eventually that the ℓ_0 th neighborhood of v in D_{ℓ_0} has size very close to linear: in particular, large enough that at the next application of Lemma 3.1 part 1 to bound the size of the $(\ell_0 + 1)$ st neighborhood of v in \mathcal{C} we will find that the second term in the minimum is smaller. It is important that this neighborhood's size should not depend upon t , and therefore at this step the minimum degree input to Lemma 3.1 must be independent of t : to achieve this we will need that \mathcal{C} contains a large fraction (independent of t) of the entire cluster graph. In a final application of Lemma 3.1 we will find a $(\ell_0 + 2)$ nd neighborhood of v in B , and this will cover a fraction of B independent of t , and thus much bigger than ε . By the same procedure we find a large $(\ell_0 + 2)$ nd neighborhood of v in B' : by (ε, p) -regularity of BB' the bipartite graph between the two $(\ell_0 + 2)$ nd neighborhoods contains many edges, and in particular paths of all (not too long) lengths. By construction a path of length $2i - 1$ vertices in this graph extends to a cycle (through v) on $2i + 2\ell_0 + 3$ vertices, as required. This approach is illustrated in the figure below.

For the proof, we choose constants as follows. Let C be the constant returned by Lemma 3.1. We will choose ε_0 to be a small enough positive function of C, d, ℓ_0, τ and α in all the calculations that follow. Then we choose n_0 to be large enough relative to these constants, ε and their reciprocals.

Let H' be the subgraph of R consisting of edges of R which form triangles together with A : by definition H' has at least τt^2 edges, and so average degree at least $2\tau t$. It follows that H' has a subgraph H whose minimum degree is at least τt , which we fix.

We will now choose sequentially the clusters required for the structure described above, under the assumption that every cluster has degree at least $\tau t/2$ in H' to unused clusters. Let B and B' be any

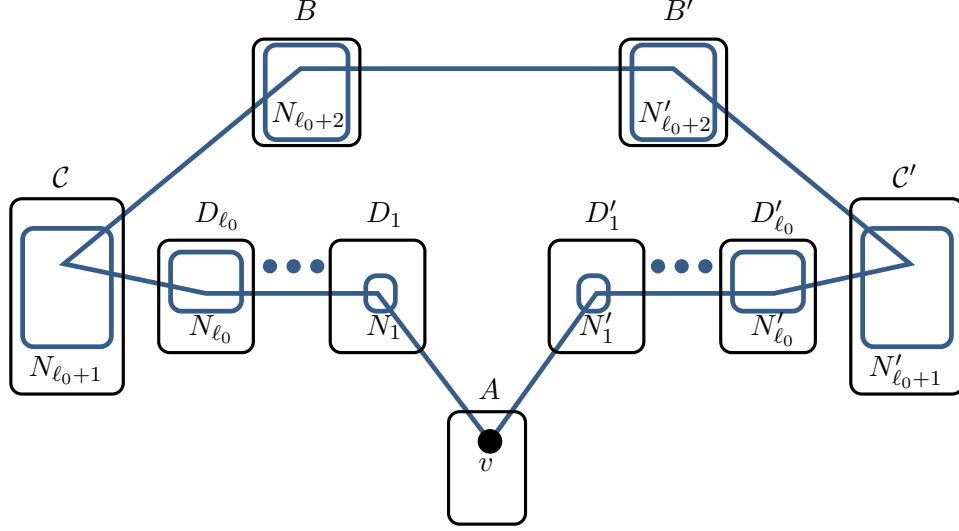


Figure 1: Constructing an odd cycle from the cluster graph

two adjacent clusters in H . Let \mathcal{C}_1 consist of $\tau t/4$ neighbors of B in H . Each of these clusters has at least $\tau t/4$ neighbors not equal to B or B' or to clusters in \mathcal{C}_1 , and consequently there is a cluster D_{ℓ_0} adjacent to a set of $\tau^2 t/16$ clusters \mathcal{C} of \mathcal{C}_1 . We now construct the path D_{ℓ_0}, \dots, D_1 greedily choosing any so far unused neighbor of each D_i in turn. We now repeat the same process (avoiding previously used clusters) to construct \mathcal{C}' and D'_{ℓ_0}, \dots, D'_1 . Note that at the end we have used only $2(\ell_0 + 1) + \tau^2 t/8 < \tau t/2$ clusters (where the inequality follows from $t \geq \varepsilon^{-1}$ and choice of ε_0), so the assumption of degree at least $\tau t/2$ to unused clusters is valid.

We now delete vertices from these clusters as follows (in a slight abuse of notation, we will continue to use the same letters for the clusters after deletion). Let

$$\delta = \tau^2 d/64 \quad \text{and} \quad \tilde{\delta} = d/4t.$$

From each cluster in \mathcal{C} we remove all vertices whose degree into B is smaller than $\tilde{\delta} n^{\alpha-1}$. Then from D_{ℓ_0} we remove all vertices whose degree into $\bigcup \mathcal{C}$ is smaller than $\delta n^{\alpha-1}$, and then for each $\ell_0 - 1 \geq i \geq 1$ in succession we remove from D_i all vertices whose degree into D_{i+1} is smaller than $\tilde{\delta} n^{\alpha-1}$. Since we have $(d - \varepsilon) > d/2$, by (ε, p) -regularity of the various pairs (including, by Proposition 2.3, the pair $(D_{\ell_0}, \bigcup \mathcal{C})$) we do not delete more than $\varepsilon n/t$ vertices (and in particular, since $\varepsilon < 1/2$, not more than half of the vertices) from any cluster. We perform the same process on the clusters $\mathcal{C}', D'_{\ell_0}, \dots, D'_1$.

We choose a vertex $v \in A$ whose neighborhood in both D_1 and D'_1 has size at least $(d - \varepsilon)n^{\alpha-1}/t > \tilde{\delta} n^{\alpha-1}$: again this is possible by (ε, p) -regularity of AD_1 and AD'_1 , and since $\varepsilon < 1/2$. Let N_1 be the neighbors of v in D_1 , and for each $2 \leq i \leq \ell_0$ let N_i be the neighbors of N_{i-1} in D_i . Let N_{ℓ_0+1} be the set of neighbors of N_{ℓ_0} in $\bigcup \mathcal{C}$, and finally N_{ℓ_0+2} be the set of neighbors of N_{ℓ_0+1} in B .

For $i \geq 0$ and $\beta \geq 1$, define $f(i, \beta) = \frac{1}{\beta-1} + \frac{\beta^2-2\beta}{(\beta-1)\beta^i}$. Note that $f(i, \beta)$ is well defined since $\beta^i + \beta^2 - 2\beta$ is divisible by $\beta - 1$ as a polynomial in β . Also, by definition, $f(1, \beta) = 1$ and $f(i, 1) = i$ for all $i \geq 0$. Let $\tilde{\gamma} = \tilde{\delta}/2C$, where C is the constant in Lemma 3.1. We claim that for each $1 \leq i \leq \ell_0$ we have

$$|N_i| \geq \min\{(\tilde{\gamma} n^{\alpha-1})^{f(i, \beta)}, \tilde{\gamma}^{1/(\alpha-1)} n\}.$$

Proceed by induction on i . The statement is true for $i = 1$ by choice of v . If the claim holds for some $1 \leq i \leq \ell_0 - 1$, then we apply Lemma 3.1 to the bipartite graph spanned by edges between $U = N_i$ and $V = N_{i+1}$. By construction, every vertex of U has degree at least $\tilde{\delta}n^{\alpha-1}$. By Lemma 3.1 part 1, if $|N_i| < |N_{i+1}|$ then

$$|N_{i+1}| \geq \min\{\tilde{\gamma}^{1/\beta}|N_i|^{1/\beta}n^{(\alpha-1)/\beta}, \tilde{\gamma}^{1/(\alpha-1)}n\}$$

provided $n \geq n_0 \geq (2C/\tilde{\delta})^{1/(\alpha-\beta)}$. By the above lower bound on $|N_i|$, the first expression in the minimum is at least

$$\tilde{\gamma}^{\frac{1}{\beta-1} + \frac{\beta^2-2\beta}{(\beta-1)\beta^{i+1}}} n^{(\alpha-1)\left(\frac{1}{\beta-1} + \frac{\beta^2-2\beta}{(\beta-1)\beta^{i+1}}\right)},$$

as required. By Lemma 3.1 part 2, if $|N_i| \geq |N_{i+1}|$ then $|N_{i+1}| \geq \tilde{\gamma}^{1/(\alpha-1)}n$, as required. This verifies the claim.

We now apply Lemma 3.1 to the bipartite graph spanned by edges between $U = N_{\ell_0}$ and $V = N_{\ell_0+1}$ to find bounds on the size of N_{ℓ_0+1} . Note that the degree of all vertices of N_{ℓ_0} into $\bigcup \mathcal{C}$ is at least $\delta n^{\alpha-1}$, by construction. We have that for $\beta = 1$, $f(\ell_0 + 1, 1) = \ell_0 + 1 > 1/(\alpha - 1)$. Also for $\beta > 1$, the definition of ℓ_0 ensures

$$f(\ell_0 + 1, \beta) = \frac{1}{\beta - 1} \left(1 - \frac{2\beta - \beta^2}{\beta^{\ell_0+1}}\right) > \frac{1}{\beta - 1} \left(1 - \frac{2\beta - \beta^2}{\frac{(2\beta - \beta^2)(\alpha-1)}{\alpha-\beta}}\right) = \frac{1}{\alpha - 1}.$$

Let $\gamma = \delta/2C$. Then Lemma 3.1 shows $|N_{\ell_0+1}| \geq \gamma^{1/(\alpha-1)}n$. Indeed, we have that the minimum in part 1 of this lemma takes the value $\gamma^{1/(\alpha-1)}n$, since $f(\ell_0 + 1, \beta) > 1/(\alpha - 1)$ and n_0 is large enough. By choosing ε_0 small enough and since $t \geq 1/\varepsilon$, we have that $\gamma^{1/(\alpha-1)}n > n/t$. Thus we are guaranteed to have $|N_{\ell_0+1}| > |B| \geq |N_{\ell_0+2}|$. Since every vertex of N_{ℓ_0+1} has at least $\tilde{\delta}n^{\alpha-1}$ neighbors in B , a final application of Lemma 3.1 part 2 to the bipartite graph spanned by edges between N_{ℓ_0+1} and N_{ℓ_0+2} with $\tilde{\gamma} = \tilde{\delta}/2C$ yields

$$|N_{\ell_0+2}| \geq \tilde{\gamma}n^{\alpha-1} \cdot \gamma^{\frac{2-\alpha}{\alpha-1}}n^{2-\alpha} = \frac{d}{8Ct}n^{\alpha-1} \cdot \left(\frac{\tau^2 d}{128C}\right)^{\frac{2-\alpha}{\alpha-1}}n^{2-\alpha} = \frac{d}{8C} \left(\frac{\tau^2 d}{128C}\right)^{\frac{2-\alpha}{\alpha-1}}n/t > \varepsilon n/t$$

where the last inequality follows by choosing ε_0 small enough.

We define similarly sets $N'_1, \dots, N'_{\ell_0+2}$ and obtain that N'_{ℓ_0+2} contains at least $\varepsilon n/t$ vertices of B' . By (ε, p) -regularity the bipartite graph $G[N_{\ell_0+2}, N'_{\ell_0+2}]$ has average degree at least $\varepsilon(d - \varepsilon)n^{\alpha-1}/t$ and so has a subgraph with minimum degree at least $\varepsilon \delta n^{\alpha-1}/4t$. This subgraph contains paths of every odd length up to $\varepsilon \delta n^{\alpha-1}/4t$. Since by our construction a path of length $2i - 1$ between B and B' extends to an odd cycle in G through v of length $2i + 2\ell_0 + 3$, this completes the proof. \square

Remark. Note that, by definition, we have that $\ell_0 \geq 1$ for all values of $1 \leq \beta < \alpha < 2$. Therefore this embedding lemma cannot be used to find odd cycles of length shorter than seven.

3.2 Expansion and embedding: complete bipartite graphs

The purpose of this section is to show that when $\mathcal{F} = \{K_{s,t}\}$ for some $2 \leq s \leq t$, if $K_{s,t}$ is $(2 - 1/s, \beta)$ -smooth for some $\beta < 2 - 1/s$, then we can improve on the expansion and embedding of the previous subsection, and embed cycles of length at least five. First we prove the necessary expansion.

Lemma 3.3. *Let G be an n -vertex $K_{s,t}$ -free graph. Let v be a vertex of G , $X \subset N(v)$, and Y a subset of $V(G)$ disjoint from $X \cup \{v\}$. Suppose that $|X| \geq \rho_X n^{1-1/s}$ and $|N(x) \cap Y| \geq \rho_Y |Y| n^{-1/s}$ for each $x \in X$. Then*

$$|N(X) \cap Y| \geq \left(\frac{\rho_X}{t-1} \right)^{\frac{1}{s-1}} (\rho_Y |Y| - sn^{1/s}).$$

Proof. Each vertex in X contains at least $\binom{\rho_Y |Y| n^{-1/s}}{s-1}$ subsets of $N(X) \cap Y$ of size $s-1$ in its neighborhood. However, if there were t vertices of X all of whose neighborhoods contained one particular $(s-1)$ -set in $N(X) \cap Y$, then together with v we would find $K_{s,t}$ in G , which is forbidden. It follows that we have

$$|X| \binom{\rho_Y |Y| n^{-\frac{1}{s}}}{s-1} \leq (t-1) \binom{|N(X) \cap Y|}{s-1}$$

and hence

$$|X| (\rho_Y |Y| n^{-\frac{1}{s}} - s + 2)^{s-1} \leq (t-1) (|N(X) \cap Y|)^{s-1}$$

which implies

$$\begin{aligned} |N(X) \cap Y| &\geq (t-1)^{-\frac{1}{s-1}} \rho_X^{\frac{1}{s-1}} n^{\frac{1}{s}} (\rho_Y |Y| n^{-\frac{1}{s}} - s + 2) \\ &\geq \left(\frac{\rho_X}{t-1} \right)^{\frac{1}{s-1}} (\rho_Y |Y| - sn^{1/s}). \end{aligned}$$

This proves Lemma 3.3. □

Our embedding statement is then the following.

Lemma 3.4. *Let $\tau, d > 0$, and suppose that for some $2 \leq s \leq t$ the graph $K_{s,t}$ is $(2 - 1/s, \beta)$ -smooth for some $\beta < 2 - 1/s$. Let $p = p(n) = n^{-1/s}$ for $n \in \mathbb{N}$. Then there exist $\varepsilon_0 = \varepsilon_0(\tau, d)$ and $n_0 = n_0(\tau, d)$ such that the following holds. If $\varepsilon < \varepsilon_0$, $n > n_0$, and G is a $K_{s,t}$ -free graph with an (ε, d, p) -cluster graph R with $q \geq \varepsilon^{-1}$ clusters in which one cluster A is in at least τq^2 triangles, then G contains C_k for each odd $k \in \mathbb{N}$ satisfying*

$$5 \leq k \leq \varepsilon d n^{1-1/s} / 4q.$$

Sketch of proof. We follow the notation and proof of Lemma 3.2, to produce two adjacent clusters B, B' and two families of sets \mathcal{C} and \mathcal{C}' disjoint from them and from each other such that the following holds. Every vertex of every set in \mathcal{C} (\mathcal{C}') has at least $dn^{1-1/s}/4q$ neighbors in B (B' respectively), each family \mathcal{C} and \mathcal{C}' contains $\tau q/4$ sets, and every set in $\mathcal{C} \cup \mathcal{C}'$ was obtained from a partition cluster, which is adjacent to A in the cluster graph, by removing fewer than $\varepsilon n/q$ vertices. In particular, every set in $\mathcal{C} \cup \mathcal{C}'$ forms together with A a $(3\varepsilon, p)$ -regular pair of relative density at least $d - \varepsilon$.

Now we slightly modify the proof. Choose a vertex $v \in A$ which has at least $\frac{\tau^2 d}{16} n^{1-1/s}$ neighbors in each of $\bigcup \mathcal{C}$ and $\bigcup \mathcal{C}'$. This vertex exists by $(3\varepsilon, p)$ -regularity of the pairs $(A, \bigcup \mathcal{C})$, $(A, \bigcup \mathcal{C}')$, and since $\mathcal{C}, \mathcal{C}'$ each contain $\tau q/4$ sets and $\varepsilon = o(d)$. We let N_1 be the neighborhood of v in $\bigcup \mathcal{C}$, and N_2 the neighborhood of N_1 in B . By Lemma 3.3 we have

$$|N_2| \geq \left(\frac{\tau^2 d}{16(t-1)} \right)^{\frac{1}{s-1}} \left(\frac{d}{4} |B| - sn^{1/s} \right) > \varepsilon |B|$$

where the last inequality comes from choosing ε_0 small enough and n_0 large enough. Similarly we define N'_1 and N'_2 and obtain that $|N'_2| > \varepsilon|B'|$. Now since B and B' form an edge of the cluster graph, the graph $G[N_2, N'_2]$ has average degree at least $\varepsilon(d - \varepsilon)n^{1-1/s}/q$. Then it has a subgraph with minimum degree at least $\varepsilon dn^{1-1/s}/4q$ and therefore contains paths of every odd length up to $\varepsilon dn^{\alpha-1}/4q$. Since by our construction a path of length $2i - 1$ between N_2 and N'_2 extends to an odd cycle in G through v of length $2i + 3$, this completes the proof. \square

4 Transference of density

Suppose that G is an \mathcal{F} -free graph, and R an (ε, d, p) -cluster graph for G . The aim of this section is to show that under the assumption of smoothness, the density of G (relative to the extremal \mathcal{F} -free graphs) transfers to an (absolute) density of R : this allows us to use classical extremal results to prove our main theorems.

Lemma 4.1. *Let \mathcal{F} be an (α, β) -smooth family with relative density ρ for some $2 > \alpha > \beta \geq 1$ and $\rho > 0$. For each $\gamma > 0$ there exist $\varepsilon_0, d_0 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_0$, $0 < d \leq d_0$ and T there is n_0 with the following property. Suppose $n \geq n_0$, G is an \mathcal{F} -free graph on n vertices, $p = n^{\alpha-2}$ and R is an (ε, d, p) -cluster graph with $\varepsilon^{-1} \leq t \leq T$ vertices as obtained from Lemma 2.2.*

1. *If $|E(G)| \geq (\mu^{\alpha-1} + \gamma)\rho p \frac{n^2}{2}$, then $|E(R)| \geq (\mu - \gamma)\frac{t^2}{2}$.*
2. *If a cluster $V_i \in V(R)$ meets at least $(\mu^{\alpha-1} + \gamma)\rho n^{\alpha-1}|V_i|$ edges of G , then either V_i is in at least γt irregular pairs, or $d_R(V_i) \geq (\mu - \gamma)t$.*

Proof. We choose $d_0 = \gamma\rho/2$, ε_0 to be sufficiently small function of γ, ρ, α and we will require n_0 to be sufficiently large for all the calculations that follow.

Let $n \geq n_0$, $0 < d \leq d_0$ and $0 < \varepsilon \leq \varepsilon_0$, and let G be an n -vertex \mathcal{F} -free graph with $(\mu^{\alpha-1} + \gamma)\rho p \frac{n^2}{2}$ edges. Let R be an (ε, d, p) -cluster graph for G with $p = n^{\alpha-2}$, as obtained from a partition \mathcal{P} given by Lemma 2.2, with clusters V_1, V_2, \dots, V_t of size m , exceptional cluster V_0 , and $1/\varepsilon \leq t \leq T(\varepsilon)$. For $i \in [t]$, let I_i and D_i respectively denote the set of irregular pairs including V_i and the set of regular pairs of density at least d including V_i . Thus $|D_i|$ is the degree of V_i in R . Let $n_i = m|D_i| + m|I_i|$. Applying the smoothness of \mathcal{F} to the bipartite subgraph H of G consisting of edges in pairs $(V_i, V_j) \in I_i \cup D_i$, we obtain

$$|E(H)| \leq z(m, n_i, \mathcal{F}) \leq \rho m n_i^{\alpha-1} + O(n_i^\beta).$$

Since $\beta < \alpha$, provided n is large enough, this is at most $(\rho + \varepsilon)m n_i^{\alpha-1}$. Therefore the total number of edges of G meeting V_i is at most

$$e(V_i, V_0) + e(V_i) + t \cdot d p m^2 + (\rho + \varepsilon)m n_i^{\alpha-1} \leq \gamma \rho m n^{\alpha-1} + \rho m n_i^{\alpha-1}. \quad (4)$$

Here we used that $e(V_i) \leq 2\rho m^\alpha \leq 2\rho m(\varepsilon n)^{\alpha-1}$, $e(V_i, V_0) \leq \rho m(\varepsilon n)^{\alpha-1} + O((\varepsilon n)^\beta) \leq 2\rho m(\varepsilon n)^{\alpha-1}$, n is large enough, ε_0 is sufficiently small, $p = n^{\alpha-2}$ and $d \leq d_0 = \gamma\rho/2$.

Summing over all clusters, we obtain

$$2|E(G)| \leq \gamma \rho m t n^{\alpha-1} + \rho m \sum_{i=1}^t n_i^{\alpha-1} \leq \gamma \rho n^\alpha + \rho m t \left(\sum_{i=1}^t \frac{n_i}{t} \right)^{\alpha-1}$$

where the final part of the inequality is Jensen's inequality, using $\alpha - 1 < 1$. On the other hand, $|E(G)| \geq (\mu^{\alpha-1} + \gamma)\rho p \frac{n^2}{2} = (\mu^{\alpha-1} + \gamma)\rho \frac{n^\alpha}{2}$. Putting these together and simplifying we have $\mu n \leq \sum_{i=1}^t \frac{n_i}{t}$, and thus

$$\sum_{i=1}^t (|D_i| + |I_i|) \geq \mu t n / m \geq \mu t^2.$$

Since \mathcal{P} is a regular partition and $\varepsilon = o(\gamma)$, the $|I_i|$ sum to at most $2\varepsilon t^2 < \gamma t^2$, and since the $|D_i|$ sum to $2|E(R)|$, part 1 follows.

For part 2, we have that the number of edges meeting V_i is at least $(\mu^{\alpha-1} + \gamma)\rho n^{\alpha-1}m$. This, together with (4), implies that $n_i \geq \mu n \geq \mu m t$. Thus $|D_i| + |I_i| \geq \mu t$, and so either $|I_i| \geq \sqrt{\varepsilon}t$ or (since $\varepsilon = o(\gamma)$) $|D_i| \geq (\mu - \gamma)t$, as desired. \square

5 Proof of Theorems 1.3 and 1.5

The extremal result which we transfer to the \mathcal{F} -free setting is essentially the Stability Theorem of Simonovits [21] for triangles. However we require a slight strengthening: we forbid the existence of a vertex contained in many triangles, rather than any triangle at all.

Lemma 5.1. *Given $0 < \gamma < 1/8$, let G be an n -vertex graph with $e(G) \geq (\frac{1}{4} - \gamma)n^2$. Then either there is a vertex of G contained in at least γn^2 triangles, or we can delete at most $9\gamma^{1/4}n^2$ edges of G to make it bipartite.*

Proof. Let u be a vertex of maximum degree in G , X be its neighborhood and let $Y = V(G) \setminus X$. By definition of u , every vertex in Y is incident with at most $|X|$ edges. Therefore the number of edges in G satisfies

$$(\frac{1}{4} - \gamma)n^2 \leq e(G) \leq e(X) + |Y||X| = e(X) + (n - |X|)|X|.$$

Thus either $e(X) \geq \gamma n^2$, in which case we are done, or we have $(\frac{1}{2} - \sqrt{2\gamma})n \leq |X| \leq (\frac{1}{2} + \sqrt{2\gamma})n$.

If $e(Y) \leq (9\gamma^{1/4} - \gamma)n^2$, then $G[X, Y]$ is the desired large bipartite subgraph of G . So suppose this is not the case. Let v be a vertex of X with maximum degree in Y , and Z its neighborhood in Y . Either $e(Z) \geq \gamma n^2$, in which case we are done, or we have

$$(9\gamma^{1/4} - \gamma)n^2 \leq e(Y) \leq \binom{|Y|}{2} - \binom{|Z|}{2} + \gamma n^2.$$

From this, using that $|Y| + |Z| \leq 2|Y| \leq (1 + 2\sqrt{2\gamma})n$, we obtain

$$(18\gamma^{1/4} - 4\gamma)n^2 \leq |Y|(|Y| - 1) - |Z|(|Z| - 1) \leq |Y|^2 - |Z|^2 \leq (1 + 2\sqrt{2\gamma})n(|Y| - |Z|).$$

Therefore $|Y| - |Z| \geq (18\gamma^{1/4} - 4\gamma)n/2 \geq 7\gamma^{1/4}n$. Since we have $|Y| \leq (\frac{1}{2} + \sqrt{2\gamma})n$, we have

$$|Z| \leq (\frac{1}{2} - 5\gamma^{1/4})n. \tag{5}$$

On the other hand, since no vertex of X has more than $|Z|$ neighbors in Y , we have

$$(\frac{1}{4} - \gamma)n^2 \leq e(G) \leq e(X) + |X||Z| + e(Y) \leq 2\gamma n^2 + |X||Z| + \binom{|Y|}{2} - \binom{|Z|}{2}.$$

Together with $|X|, |Y| \leq (\frac{1}{2} + \sqrt{2\gamma})n$ and $|Z| \leq n$, this implies that

$$(\frac{1}{4} - 3\gamma)n^2 \leq (\frac{1}{2} + \sqrt{2\gamma})n|Z| + \frac{(\frac{1}{2} + \sqrt{2\gamma})^2 n^2}{2} - \frac{|Z|^2}{2} \leq \sqrt{2\gamma}n^2 + \frac{(\frac{1}{2} + \sqrt{2\gamma})^2 n^2}{2} + \frac{(n - |Z|)|Z|}{2}.$$

Simplifying, we get $(\frac{1}{4} - 3\sqrt{2\gamma} - 8\gamma)n^2 \leq (n - |Z|)|Z|$. Finally, this implies

$$|Z| \geq (\frac{1}{2} - \sqrt{3\sqrt{2\gamma} + 8\gamma})n \geq (\frac{1}{2} - 4\gamma^{1/4})n,$$

which contradicts (5) and completes the proof. \square

We will in fact prove slightly more than the statement of Theorem 1.3: we will show that any family of $\mathcal{F} \cup \{C_k\}$ -free graphs in which the number of edges is asymptotically extremal, is near-bipartite (rather than just proving it for the extremal graphs themselves).

Proof of Theorem 1.3. Let \mathcal{F} be an (α, β) -smooth family and let $p = n^{\alpha-2}$. By definition, we have that the maximum possible number of edges in a \mathcal{F} -free bipartite graph on n vertices is at most $(1 + o(1))\rho(n/2)^\alpha = ((1/2)^{\alpha-1} + o(1))\rho p \frac{n^2}{2}$. Let $\gamma > 0$ be an arbitrarily small but fixed constant and let $\varepsilon_0 \leq \gamma$ and $d \leq d_0 = \gamma\rho/2$ be sufficiently small for Lemma 4.1. Let $\varepsilon \leq \varepsilon_0$ be sufficiently small for Lemma 3.2 with

$$\tau := \frac{1}{2} - \frac{1}{2}(1 - 2^\alpha \gamma)^{1/(\alpha-1)} + \gamma.$$

Note that as $\gamma > 0$ tends to zero, by definition $\tau > 0$ also tends to zero. Let T be the constant returned by Lemma 2.2, and n_0 be large enough for Lemmas 3.2 and 4.1.

Let $n \geq n_0$, and G be an n -vertex \mathcal{F} -free graph with

$$e(G) \geq \left(\left(\frac{1}{2}\right)^{\alpha-1} - \gamma\right)\rho p \frac{n^2}{2} = \left(\left(\frac{1}{2} - \tau + \gamma\right)^{\alpha-1} + \gamma\right)\rho p \frac{n^2}{2},$$

where the equality comes from the definition of τ .

By Lemma 2.2 there is a sparse-regular partition of G with a t -vertex (ε, d, p) -cluster graph R , where $\varepsilon^{-1} \leq t \leq T$. By Lemma 4.1 (with $\mu = 1/2 - \tau + \gamma$) we have $e(R) \geq (\frac{1}{2} - \tau)\frac{t^2}{2} \geq (\frac{1}{4} - \tau)t^2$. If R has a vertex in τt^2 triangles, then by Lemma 3.2 G contains C_k for each $k \geq k_0 = 2\ell_0 + 5$ with ℓ_0 specified in the statement of Lemma 3.2, and we are done. If not, then by Lemma 5.1 R has a subgraph R' with at most $9\tau^{1/4}t^2$ edges such that $E(R) \setminus E(R')$ is bipartite.

Let G' be the spanning subgraph of G consisting of edges belonging to pairs in $E(R')$, pairs with density less than d , irregular pairs and edges inside partition classes. Note that the (ε, p) -regular partition of G which gives the (ε, d, p) -cluster graph R is also an (ε, p) -regular partition of G' , and it gives the graph R' as an (ε, d, p) -cluster graph for G' . Since $|E(R')| \leq 9\tau^{1/4}t^2$, it follows by Lemma 4.1, with $\mu = 9\tau^{1/4} + \gamma$, that we have $e(G') \leq ((9\tau^{1/4} + \gamma)^{\alpha-1} + \gamma)\rho p \frac{n^2}{2}$. Since $\tau \rightarrow 0$ as $\gamma \rightarrow 0$, this shows that $e(G') = o(e(G))$ and since the graph with edges $E(G) \setminus E(G')$ is bipartite, this proves that G is near-bipartite. \square

The proof of Theorem 1.5 is identical except that Lemma 3.4 replaces Lemma 3.2.

6 Proof of Theorem 1.4

We have shown that the extremal graphs for forbidding \mathcal{F} and C_k , for any smooth \mathcal{F} , are near-bipartite. In this section we prove Theorem 1.4, which says indeed that extremal graphs are actually bipartite provided they have large enough minimum degree and sufficiently many vertices.

Sketch of proof. Write $a(n) \ll b(n)$ for $a(n) = o(b(n))$ as $n \rightarrow \infty$. Fix constants $0 < \varepsilon \ll \varepsilon' \ll d \ll \tau \ll \gamma \ll 1$ so that all inequalities required below are satisfied (for the purposes of this sketch we will not specify the constants). Let $\ell_0 = \ell_0(\alpha, \beta)$ be the constant defined in Lemma 3.2. Let $k_0 = 2\ell_0 + 3$ and suppose $k \geq k_0$ is odd. We assume n to be sufficiently large given the preceding constants, and set $p = n^{\alpha-2}$. Let G be an n -vertex \mathcal{F} -free graph with minimum degree at least $\rho(\frac{2n}{5} + \gamma n)^{\alpha-1}$. Let R be the associated t -vertex (ε, d, p) -cluster graph generated by Lemma 2.2. By Lemma 4.1, all but at most $\varepsilon't$ clusters of R have degree at least $(2/5 + \gamma/2)t$. Also, one can show that there are at most $\varepsilon't$ clusters that are ‘bad’, in that they contain more than $\varepsilon'n/t$ vertices that are incident to more than $2dpn$ edges in pairs that are irregular or have relative density less than d . Thus we may restrict to a subgraph R' of R with at least $(1 - 2\varepsilon')t$ clusters, all of which are good, such that the minimum degree in R' is at least $(2/5 + \gamma/4)t$. Now Lemma 9 in [1], which is a ‘supersaturated version’ of the Andrásfai-Erdős-Sós theorem, guarantees that either there is a vertex of R' contained in τt^2 triangles, or there is a partition $(\mathcal{X}, \mathcal{Y})$ of $V(R')$ such that for each vertex of R' , all but $\gamma t/16$ of its incident edges cross $(\mathcal{X}, \mathcal{Y})$. In the former case the proof is similar to that of Theorem 1.3, so we may suppose we are in the latter case. Also note that $2t/5 \leq |\mathcal{X}|, |\mathcal{Y}| \leq 3t/5$.

Let (X, Y) be a maximal cut of G obtained from the partition $(\bigcup \mathcal{X}, V(G) \setminus \bigcup \mathcal{X})$ by successively moving vertices with a majority of neighbors in their own partition class to the other partition class. Using regularity and the fact that all clusters in R' are good, one can check that at most a $2\varepsilon'$ -fraction of any cluster in $V(R')$ will be moved in this process. By maximality, each vertex of G is incident to at least $\frac{1}{2}\rho(\frac{2n}{5})^{\alpha-1}$ edges crossing (X, Y) .

Since G is not bipartite, suppose without loss of generality that uv is an edge in X . Let N_1 and N'_1 be disjoint, equal-sized sets of neighbors of respectively u and v in Y . Let N_2 and N'_2 be disjoint, equal-sized sets of neighbors of respectively N_1 and N'_1 in $X \setminus u, v$, and so on. Using Lemma 3.1 and similar computation as in the proof of Lemma 3.2, it is not hard to check that it is possible to choose these sets in such a way that after $\ell = \ell_0 + 1$ steps we have sets N_ℓ and N'_ℓ which are both of size τn and are disjoint. Let $\mathcal{C}, \mathcal{C}'$ respectively be the set of clusters C that contain at least $\tau|C|/2$ vertices of N_ℓ, N'_ℓ . Note that $|\mathcal{C}| \geq (|N_\ell| - \tau n/2)/(n/t) = \tau t/2$, and similarly $|\mathcal{C}'| \geq \tau t/2$. Fix any $C \in \mathcal{C}$. Since each vertex of R' has at most $\gamma t/16$ neighbors in its own part, for any $C' \in \mathcal{C}'$ there are at least $(2/5 + 3\gamma/16)t + (2/5 + 3\gamma/16)t - 3t/5 \geq t/5$ clusters that are common neighbors of C and C' . Since $|\mathcal{C}'| \geq \tau t/2$, a simple double counting argument shows that in particular there is a cluster D of R' which is adjacent both to C and to a set \mathcal{D} of at least $\tau t/10$ clusters of \mathcal{C}' . Let Z be obtained from $N'_\ell \cap \bigcup_{C' \in \mathcal{D}} C'$ by deleting all vertices with degree less than $dn/4t$ in D . Then $|Z| \geq \tau^3 n$. Applying Lemma 3.1 part 2 between Z and D , we reach a set $N'_{\ell+1}$ in D disjoint from all previous N_i and N'_i , and of size exceeding $\varepsilon|D|$. By (ε, p) -regularity of the pair (C, D) , there is a path between $N_\ell \cap C$ and $N'_{\ell+1}$ of length $k - 2\ell - 2$. By construction, this can be extended to a k -cycle in G . \square

The value of k_0 in this theorem is similar in size to that in Theorem 1.3. In the case of the complete bipartite graphs $K_{2,t}$ and $K_{3,3}$ (with more care in the argument) we can have $k_0 = 5$. Finally, if

we impose the further condition $e(G) \sim \rho(n/2)^\alpha$, then we can reduce the minimum degree to $o(\rho pn)$ (in this case Theorem 1.3 shows that the cluster graph of G is almost bipartite). This reduces the problem of showing that extremal n -vertex $\mathcal{F} \cup C_k$ -free graphs are bipartite to that of showing that they do not have vertices of degree much smaller than the average. Unfortunately, even for $\mathcal{F} = \{C_4\}$ this is not known for most values of n .

7 A new smooth family

Let B_t be the book with t pages: that is, t copies of $K_{2,2}$ sharing a common edge (and no other vertices). In this section, we give an example of an application of Theorem 1.3 which determines the asymptotic behavior of the Turán numbers $\text{ex}(n, \{K_{2,t}, B_t, C_k\})$ for $k \geq 9$. For the proof, we require one result on counting 4-cycles in graphs, which will be used to show $\{K_{2,t}, B_t\}$ is a smooth family.

Lemma 7.1. *Let G be a bipartite graph with parts U and V of sizes m and n and e edges, and suppose that $e(e-n) \geq nm(m-1)/2$. Then the number of 4-cycles in G is at least*

$$\frac{e^2(e-n)^2 - e(e-n)nm(m-1)}{4n^2m(m-1)}.$$

Proof. Let $N := \sum_{v \in V} \binom{d(v)}{2}$. Note that $N = \sum_{\{u,v\} \subset U} d(u,v)$, where $d(u,v)$ is the number of common neighbors of u and v . The number of 4-cycles in G is exactly $\sum_{\{u,v\} \subset U} \binom{d(u,v)}{2}$. By Jensen's inequality, this is at least

$$\frac{m(m-1)}{2} \left(\frac{\frac{2}{m(m-1)} \sum_{\{u,v\} \subset U} d(u,v)}{2} \right) = \frac{m(m-1)}{2} \left(\frac{\frac{2N}{m(m-1)}}{2} \right) = \frac{N^2}{m(m-1)} - \frac{N}{2}.$$

Applying Jensen's inequality a second time, we have

$$N \geq n \binom{e/n}{2} = \frac{e(e-n)}{2n}.$$

Since $e(e-n) \geq nm(m-1)/2$, we deduce $N \geq m(m-1)/4$. For these values of N , $N^2/m(m-1) - N/2$ is non-decreasing, and therefore the number of 4-cycles in G is at least

$$\frac{N^2}{m(m-1)} - \frac{N}{2} \geq \frac{e^2(e-n)^2 - e(e-n)nm(m-1)}{4n^2m(m-1)}.$$

□

Now we show that $\{K_{2,t}, B_t\}$ is smooth.

Theorem 7.2. *For each $t \geq 2$, the family $\{K_{2,t}, B_t\}$ is $(\frac{3}{2}, 1)$ -smooth. Consequently, for $k \geq 9$,*

$$\text{ex}(n, \{K_{2,t}, B_t, C_k\}) \sim z(n, K_{2,t}, B_t) \sim (n/2)^{3/2}.$$

Proof. The asymptotic formula will follow from Theorem 1.3 and the known lower bound on the number of edges in a $K_{2,2}$ -free bipartite graph, namely

$$z(n, \{K_{2,t}, B_t\}) \geq z(n, K_{2,2}) \geq \left(\frac{n}{2}\right)^{3/2} - O(n).$$

Next we require an upper bound on $z(m, n, \{K_{2,t}, B_t\})$. We shall show

$$z(m, n, \{K_{2,t}, B_t\}) < mn^{1/2} + 4t^2n.$$

Let G be a $\{K_{2,t}, B_t\}$ -free bipartite graph with parts X and Y of sizes m and n , where $m \leq n$. Suppose for a contradiction, that G has at least $mn^{1/2} + 4t^2n$ edges. We may delete edges of G to obtain a subgraph H of G with exactly $e := mn^{1/2} + 4t^2n$ edges. Clearly, H is also $\{K_{2,t}, B_t\}$ -free. Let xy be any edge of H , and consider the subgraph H_{xy} of H induced by $N(x) \setminus \{y\}, N(y) \setminus \{x\}$. If this subgraph has a vertex of degree $t-1$, then H contains $K_{2,t}$; if it has a matching of t edges, then H contains B_t . It follows that since H is $\{K_{2,t}, B_t\}$ -free, then H_{xy} has at most $2(t-1)(t-2)$ edges, as a maximum matching has at most $2(t-1)$ vertices each incident to at most $t-2$ edges. Now the number of 4-cycles in H is precisely $\frac{1}{4} \sum_{xy} e(H_{xy})$, which is thus at most $(t-1)(t-2)e(H)/2$. Since

$$e(e-n) = (mn^{1/2} + 4t^2n)(mn^{1/2} + (4t^2-1)n) \geq (mn^{1/2})^2 = nm^2$$

we may apply Lemma 7.1 to conclude that the number of 4-cycles in H is at least

$$\frac{(mn^{1/2} + 3t^2n)^4 - (mn^{1/2} + 4t^2n)^2 nm^2}{4n^2 m^2} \geq t^2 mn^{1/2} + 2t^4 n > \frac{t^2}{2} (mn^{1/2} + 4t^2 n) > (t-1)(t-2) \frac{e(H)}{2},$$

which is a contradiction. This shows that $\{K_{2,t}, B_t\}$ is $(\frac{3}{2}, 1)$ -smooth. Hence we can use Theorem 1.3 (in the form it was proved in Section 5) with $k_0 = 2\ell_0 + 5$ and $\ell_0 = \lfloor 1/(\alpha-1) \rfloor = 2$ to complete the proof. \square

We remark that one could use the methods of the above proof to give new asymptotics for Turán numbers of various other families of bipartite graphs plus an odd cycle.

8 Proof of Theorem 1.6

Füredi [12] showed that

$$z(n, n, K_{2,3}) \sim \sqrt{2}n^{3/2}.$$

In this section, we describe a construction of a tripartite graph G_q with no triangle or $K_{2,3}$ having $n = 3q^2$ vertices and

$$e(G_q) = 3q^2(q-1) = \frac{1}{\sqrt{3}}n^{3/2} - n,$$

for any prime $q \equiv 2 \pmod{3}$. This construction is denser than any possible bipartite construction by a multiplicative factor of $2/\sqrt{3}$, as for n even, $z(n/2, n/2, K_{2,3}) \sim \frac{1}{2}n^{3/2}$, and so Theorem 1.6 is proved for $t = 1$. Note that the exact value $z(n, n, K_{2,3})$ is known only for finitely many n . We also remark that the purpose of choosing $q \equiv 2 \pmod{3}$ is to make -3 a non-residue mod q . This follows from the law of quadratic reciprocity, which states that if p and q are odd primes, and we define p^* to equal

p if $p \equiv 1 \pmod{4}$ or $-p$ if $p \equiv 3 \pmod{4}$, then q is a quadratic residue mod p if and only if p^* is a quadratic residue mod q . Applying this with $p = 3$ we see that -3 is a quadratic residue mod q if and only if q is a quadratic residue mod 3, i.e. $q \equiv 1 \pmod{3}$.

Construction of G_q . Let G_q have parts A_1, A_2 and A_3 which are copies of $\mathbb{F}_q \times \mathbb{F}_q$. Join $(x_1, x_2) \in A_i$ to $(y_1, y_2) \in A_{i+1}$ (where $A_4 := A_1$) if $(x_1, x_2) - (y_1, y_2) = (a, a^2)$ for some $a \in \mathbb{F}_q^*$.

Clearly G_q has $3q^2$ vertices and $3q^2(q-1)$ edges.

Claim 1. G_q is triangle-free.

Proof of Claim 1. Suppose for a contradiction that we have a triangle in G_q , with vertices $(x_1, x_2) \in A_1$, $(y_1, y_2) \in A_2$ and $(z_1, z_2) \in A_3$. Then $(x_1, x_2) - (y_1, y_2) = (a, a^2)$, $(y_1, y_2) - (z_1, z_2) = (b, b^2)$, $(z_1, z_2) - (x_1, x_2) = (c, c^2)$ for some $a, b, c \in \mathbb{F}_q^*$. It follows that $a + b + c = 0$ and $a^2 + b^2 + c^2 = 0$, so

$$0 = 2(a^2 + b^2 + c^2) - (a + b + c)^2 = (a - b)^2 + c^2 - 2(a + b)c = (a - b)^2 + 3c^2.$$

This implies that $-3 = \left(\frac{a-b}{c}\right)^2$ is a quadratic residue. But this contradicts our assumption that $q \equiv 2 \pmod{3}$, so G_q is triangle-free.

Claim 2. G_q is $K_{2,3}$ -free.

Proof of Claim 2. First we claim that each bipartite graph $G_q(A_i, A_{i+1})$ is C_4 -free. If not, then we have distinct vertices

$$(x_1, x_2), (x'_1, x'_2) \in A_i \quad (y_1, y_2), (y'_1, y'_2) \in A_{i+1}$$

and $a, b, c, d \in \mathbb{F}_q^*$ such that

$$\begin{aligned} (x_1, x_2) - (y_1, y_2) &= (a, a^2) & (x'_1, x'_2) - (y'_1, y'_2) &= (b, b^2) \\ (x_1, x_2) - (y'_1, y'_2) &= (c, c^2) & (x'_1, x'_2) - (y_1, y_2) &= (d, d^2). \end{aligned}$$

Then $a + b = c + d$ and $a^2 + b^2 = c^2 + d^2$, so

$$(a - b)^2 = 2(a^2 + b^2) - (a + b)^2 = 2(c^2 + d^2) - (c + d)^2 = (c - d)^2.$$

Now we either have $a - b = c - d$, which gives $a = c$ and $b = d$, or $a - b = d - c$, which gives $a = d$ and $b = c$. Either way we contradict the fact that we chose distinct vertices above, so $G_q(A_i, A_{i+1})$ is a C_4 -free graph. It follows that any pair of vertices in a given A_i has at most one common neighbor in each A_j with $j \neq i$. Thus in any potential $K_{2,3} \subset G_q$ the part of $K_{2,3}$ of size 2 must have vertices in different parts of G_q . Suppose, without loss of generality, that we have a vertex in A_0 and a vertex in A_1 that have 3 common neighbors in A_2 . Then we obtain distinct $a, c, e \in \mathbb{F}_q^*$ and distinct $b, d, f \in \mathbb{F}_q^*$ such that $a + b = c + d = e + f$ and $a^2 + b^2 = c^2 + d^2 = e^2 + f^2$. As above, this implies $(a - b)^2 = (c - d)^2 = (e - f)^2$. On taking square roots, each of the three terms in the identity have two possible signs. We can assume without loss of generality that the first two terms get the same sign, i.e. $a - b = c - d$. This gives $a = c$ and $b = d$, which contradicts distinctness. Therefore G_q is $K_{2,3}$ -free.

8.1 Construction of $K_{2,2t+1}$ -free graphs

Here we generalize the construction in the previous subsection. For any $t \geq 1$ and certain primes q we construct $(t+2)$ -partite graphs $G_{q,t}$ with no triangle or $K_{2,2t+1}$ having $n = (t+2)q^2$ vertices and

$$e(G_{q,t}) = \binom{t+2}{2} q^2 (q-1) = \frac{t+1}{2\sqrt{t+2}} n^{3/2} - \frac{1}{2} (t+1)n.$$

Since Füredi [12] showed that

$$z(n/2, n/2, K_{2,2t+1}) \leq \frac{\sqrt{t}}{2} n^{3/2} + n/4$$

this construction is denser than any bipartite construction by an asymptotic multiplicative factor of $(t+1)/\sqrt{t(t+2)}$, as required for Theorem 1.6.

Construction of $G_{q,t}$. Fix a prime q and ‘multipliers’ $m_{i,j} \in \mathbb{F}_q$ for $1 \leq i < j \leq t+2$ satisfying certain conditions to be described below. Let $G_{q,t}$ have parts A_1, \dots, A_{t+2} which are copies of $\mathbb{F}_q \times \mathbb{F}_q$. For $i < j$, join $(x_1, x_2) \in A_i$ to $(y_1, y_2) \in A_j$ if $(x_1, x_2) - (y_1, y_2) = m_{i,j}(c, c^2)$ for some $c \in \mathbb{F}_q^*$.

Clearly $G_{q,t}$ has $(t+2)q^2$ vertices and $\binom{t+2}{2} q^2 (q-1)$ edges. We will derive conditions on the $m_{i,j}$ which ensure that $G_{q,t}$ has no triangle or $K_{2,2t+1}$, and then show that the conditions can be satisfied. We adopt the convention that $m_{i,j} = -m_{j,i}$ when $1 \leq j < i \leq t+2$.

Claim 1. $G_{q,t}$ is triangle-free if for all distinct i, j, k ,

$$m_{i,j,k} := -m_{i,j}m_{j,k}m_{k,i}(m_{i,j} + m_{j,k} + m_{k,i})$$

is a quadratic non-residue in \mathbb{F}_q .

Proof of Claim 1. Suppose $G_{q,t}$ has a triangle with vertices $(x_1, x_2) \in A_i$, $(y_1, y_2) \in A_j$, $(z_1, z_2) \in A_k$, $a, b, c \in \mathbb{F}_q^*$ with $(x_1, x_2) - (y_1, y_2) = m_{i,j}(a, a^2)$, $(y_1, y_2) - (z_1, z_2) = m_{j,k}(b, b^2)$, $(z_1, z_2) - (x_1, x_2) = m_{k,i}(c, c^2)$. Then $m_{i,j}a + m_{j,k}b + m_{k,i}c = 0$ and $m_{i,j}a^2 + m_{j,k}b^2 + m_{k,i}c^2 = 0$, so

$$\begin{aligned} 0 &= (m_{i,j} + m_{j,k})(m_{i,j}a^2 + m_{j,k}b^2 + m_{k,i}c^2) - (m_{i,j}a + m_{j,k}b + m_{k,i}c)^2 \\ &= m_{i,j}m_{j,k}(a-b)^2 + (m_{i,j} + m_{j,k} - m_{k,i})m_{k,i}c^2 - 2(m_{i,j}a + m_{j,k}b)m_{k,i}c \\ &= m_{i,j}m_{j,k}(a-b)^2 + (m_{i,j} + m_{j,k} + m_{k,i})m_{k,i}c^2, \end{aligned}$$

and hence

$$-(m_{i,j} + m_{j,k} + m_{k,i})m_{i,j}m_{j,k}m_{k,i} = m_{i,j}^2 m_{j,k}^2 (a-b)^2 / c^2.$$

Thus we can make $G_{q,t}$ triangle-free if we arrange that $m_{i,j,k} = -m_{i,j}m_{j,k}m_{k,i}(m_{i,j} + m_{j,k} + m_{k,i})$ is a non-residue for all distinct i, j, k . (Note that $m_{i,j,k}$ does not depend on the order of i, j, k .) This proves Claim 1.

Claim 2. $G_{q,t}$ is $K_{2,2t+1}$ -free if $m_{j,k} \neq -m_{k,i}$ for all distinct i, j, k .

Proof of Claim 2. Each pair (A_i, A_j) induces a C_4 -free bipartite graph, so any pair of vertices in a given A_i has at most one common neighbor in each A_j with $j \neq i$. Thus in any potential $K_{2,2t+1}$ the part of size 2 must use two different parts, say we have $x \in A_i$ and $y \in A_j$ with $2t+1$ common

neighbors. There are t parts besides A_i and A_j , so one of them, say A_k , contains at least 3 common neighbors of x and y . Then we obtain distinct $a, c, e \in \mathbb{F}_q^*$ and distinct $b, d, f \in \mathbb{F}_q^*$ such that

$$m_{j,k}a + m_{k,i}b = m_{j,k}c + m_{k,i}d = m_{j,k}e + m_{k,i}f$$

and

$$m_{j,k}a^2 + m_{k,i}b^2 = m_{j,k}c^2 + m_{k,i}d^2 = m_{j,k}e^2 + m_{k,i}f^2.$$

Since

$$m_{j,k}m_{k,i}(a-b)^2 = (m_{j,k} + m_{k,i})(m_{j,k}a^2 + m_{k,i}b^2) - (m_{j,k}a + m_{k,i}b)^2$$

this gives $(a-b)^2 = (c-d)^2 = (e-f)^2$. On taking square roots, we can assume without loss of generality that the first two terms get the same sign, i.e. $a-b = c-d$, so $a-c = b-d$. We also have $m_{j,k}(a-c) = -m_{k,i}(b-d)$, so if $m_{j,k} \neq -m_{k,i}$ we get $a=c$ and $b=d$, which contradicts distinctness. Thus we can make $G_{q,t}$ be $K_{2,2t+1}$ -free if we arrange that $m_{j,k} \neq -m_{k,i}$ for all distinct i, j, k . This proves Claim 2.

It remains to show that we can choose multipliers $m_{i,j}$ satisfying the conditions in Claims 1 and 2. Intuitively, this should be easy for fixed t and large q by choosing random multipliers: then we will be unlikely to have $m_{j,k} = -m_{k,i}$, and given $m_{i,j}$, $m_{j,k}$, the choice of $m_{k,i}$ will be equally likely to make $m_{i,j,k}$ a residue or a non-residue. In fact, it is even possible to choose multipliers greedily to satisfy the conditions, which we shall do. To simplify our task we choose $q > 2^{t^4}$, arbitrarily relabel the multipliers as m_1, m_2, \dots, m_T , where $T = \binom{t+2}{2}$, and choose them so that each m_i is a residue, we do not have $m_i = m_j$ or $m_i = -m_j$ with $i \neq j$, and $-(m_i + m_j + m_k)$ is a non-residue for all distinct i, j, k : since the product of a residue and a non-residue is a non-residue, this will suffice.

We can easily choose residues m_1 and m_2 with $m_1 \neq m_2$ and $m_1 \neq -m_2$. Suppose inductively that we have chosen m_1, m_2, \dots, m_r satisfying these conditions for some $2 \leq r < T$. We use Weil's inequality in the following form (see, e.g., [19, Chapter 5]):

Proposition 8.1. *Suppose χ is the quadratic character, i.e. $\chi(x)$ is 1 if x is a residue, -1 if x is a non-residue, or 0 if $x = 0$. Suppose also that $f \in \mathbb{F}_q[X]$ is a polynomial of degree $d \geq 1$ that is not the square of another polynomial. Then $\left| \sum_{x \in \mathbb{F}_q} \chi(f(x)) \right| \leq (d-1)\sqrt{q}$.*

Let $I = \{-m_i - m_j : 1 \leq i < j \leq r\}$ and $g(x) = \prod_{m \in I} (1 - \chi(m - x^2))$. Note that if $m - x^2$ is a non-residue for all $m \in I$ then $g(x) = 2^{|I|}$. On the other hand, if $m - x^2$ is a residue for any $m \in I$ then $g(x) = 0$. There are at most $2|I|$ values of x such that $m - x^2$ is zero for some $m \in I$ and each of these gives a value of $g(x)$ between 0 and $2^{|I|-1}$. Let X be the set of x such that $m - x^2$ is a non-residue for all $m \in I$. Then we have

$$\left| \sum_{x \in \mathbb{F}_q} g(x) - 2^{|I|}|X| \right| \leq 2^{|I|}|I|.$$

We can also expand the product in the definition of g to get

$$\sum_{x \in \mathbb{F}_q} g(x) = q + \sum_{\emptyset \neq S \subseteq I} (-1)^{|S|} \sum_{x \in \mathbb{F}_q} \prod_{m \in S} \chi(m - x^2).$$

Applying Weil's inequality to the functions

$$f_S = \prod_{m \in S} \chi(m - x^2)$$

we have

$$\left| \sum_{x \in \mathbb{F}_q} g(x) - q \right| \leq \sum_{\emptyset \neq S \subseteq I} (2|S| - 1) \sqrt{q} \leq 2^{|I|} (2|I| - 1) \sqrt{q}.$$

We deduce that $|2^{|I|}|X| - q| \leq 2^{|I|}|I| + 2^{|I|}(2|I| - 1)\sqrt{q}$. Since $q > 2^{t^4}$ we have $|X| > 4r$, so we can choose $m_{r+1} = x^2$ for some $x \in X$ such that m_{r+1} is not equal to m_i or $-m_i$ for any $1 \leq i \leq r$. Therefore we can choose the multipliers m_1, m_2, \dots, m_T satisfying the conditions in Claims 1 and 2.

9 Concluding remarks

- In this paper we give a different and new approach to the Erdős-Simonovits conjecture, employing Scott's sparse version of Szemerédi's Regularity Lemma. This approach could potentially be used to give broader structural understanding of extremal \mathcal{F} -free graphs when \mathcal{F} consists of bipartite graphs together with graphs of chromatic number r . In the present paper, we deal with $r = 3$. More generally, let \mathcal{C}_k^r denote the family of all graphs of chromatic number r with at most k vertices, and \mathcal{C}^r denote the family of all graphs of chromatic number r . Generalizing the Erdős-Simonovits conjecture, we pose the following problem:

Problem 9.1. *Let \mathcal{F} be a family of bipartite graphs. Determine whether there exists an integer k such that*

$$ex(n, \mathcal{F} \cup \mathcal{C}_k^r) \sim ex(n, \mathcal{F} \cup \mathcal{C}^r).$$

This would suggest that the extremal density of an \mathcal{F} -free graph without any small r -chromatic subgraphs is asymptotic to the extremal density of an \mathcal{F} -free $(r - 1)$ -partite graph. A key obstacle is to find an appropriate embedding lemma analogous to Lemma 3.2.

- Exact results are rare for bipartite Turán problems. Even for C_4 , while the exact result for $z(n, n, C_4)$ is relatively straightforward, Füredi's exact result for $ex(n, C_4)$ is an intricate argument. No exact result is known for $z(n, n, K_{2,3})$ (except for finitely many n). Any construction achieving the natural upper bound would provide a positive answer to the famous biplane problem, which asks if there are infinitely many n such that there is a family A_1, \dots, A_n of subsets of an n -element set satisfying $|A_i \cap A_j| = 2$ for $i \neq j$. Suppose that there is a bipartite graph with parts X and Y of size n such that every pair of vertices in X has exactly 2 neighbors in Y . Then the set of all neighborhoods of vertices in X is a biplane on Y .

- Unsurprisingly, it is not in general easy to show that any particular family \mathcal{F} is smooth. Besides $K_{2,t}$ and $K_{3,3}$, we show that the family $\{K_{2,t}, B_t\}$ is smooth for all $t \geq 2$ (where B_t is the book of 4-cycles with t pages). We cannot at present prove, however, that B_t itself is smooth. We intend to investigate smooth families in future work.

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